λ_1, λ_2	are the thermal conductivities;
$A, B, \delta, m,$	
β, t, S_1, S_2	are the dimensionless constants;
v, w	are the boundary-layer functions;
λ_x, λ_y	are the effective thermal conductivities.

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IDENTIFICATION OF TIME-VARIABLE COEFFICIENTS OF HEAT TRANSFER BY SOLVING A NONLINEAR INVERSE PROBLEM OF HEAT CONDUCTION

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The solution of the inverse nonstationary problem of nonlinear heat conduction by using the method of optimal dynamic filtering is considered.

The solution of the inverse heat-conduction problem has lately assumed especially great importance, since one has to determine the boundary conditions of the heat transfer from the limited information on the temperature field of the body.

In [1, 2] the feasibility of electrical modelling of converse problems was considered; several approaches have been suggested for solving such problems on various analog models. With this aim in mind, the application of optimal dynamic filtering [3] is of some interest; it provides the possibility, as seen from previous investigations [4, 5], of solving a wide class of inverse heat-conduction problems, including the reconstruction of the temperature field, the determination of the boundary conditions, the restoring of the initial distributions, etc.

In this article a technique that enables one to obtain in a special way a prediction of the estimate of the state vector is proposed. The employed discrete-filtering algorithm of Kalman presupposes that an extended state vector can be estimated due to the specific shape of the solution of the inverse problem, in which side by side with the reconstruction of the temperature field, the identification of the boundary conditions is carried out. In view of the latter, the components of the temperature field vector and the identifying vector of parameters α are included in the state vector.

To construct a solution algorithm of the inverse problem a mathematical model was adopted by us in which the finite-differences equation of heat conduction in its matrix form as well as the identifying parameter α as a function of time are included:

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$$A\,(\mathbf{T}_k)\,\mathbf{X}_{k+1} = C\,(\mathbf{T}_k)\,\mathbf{X}_k + D\,(\mathbf{T}_k)\,\mathbf{U}_k + G_{k+1}\,\mathbf{W}_{k+1}; \tag{1}$$

$$\alpha = \mathbf{f}\,(\tau), \tag{2}$$

where the subscripts k refers to the k-th time instant; $A[m \times m]$, $C[m \times m]$, $D[m \times (r-l)]$, and $G[m \times p]$ are nonstationary coefficient matrices; $X[m \times 1]$ is the extended state vector which includes the identification parameter $\alpha[l \times 1]$ (T_{av} corresponding to this α is included in the coefficient matrix A); $U[(r-l) \times 1]$ is the control vector which does not include an expression with the identification parameter (αT_{av}) and $W[p \times 1]$ is the vector of random perturbations of the system input whose statistical characteristics are described by the Gaussian white noise.

Equation (1) can also the written as

$$A_1\,(\mathbf{X}_k)\,\mathbf{T}_{k+1} = C_1\,(\mathbf{X}_k)\,\mathbf{T}_k + D_1\,(\mathbf{X}_k)\,\mathbf{U}_k^1 + G_{k+1}\mathbf{W}_{k+1}, \tag{3}$$

in which, in contrast to Eq. (1) where X_k is the extended state vector, the estimated parameter α is not included in the vector of the temperature field $T[(m-l) \times 1]$, but is included in the nonstationary coefficient matrices ($A_1[(m-l) \times (m-l)]$, $C_1[(m-l)]$, $C_1[(m-l) \times (m-l)]$, $D_1[(m-l) \times r]$, and $G[(m-l) \times p]$) and in the control vector ($U^1[r \times 1]$).

Equations (1)-(3) can be transformed to the form used in the optimal dynamic filtering of the original mathematical models

$$\mathbf{X}_{k+1} = \Phi_{k+1,k}\,\mathbf{X}_k + F_{k+1,k}\,\mathbf{U}_k + G_{k+1,k}\,\mathbf{W}_k, \tag{4}$$

where $\Phi_{k+1,k}$ $[m \times m]$, $F_{k+1,k}$ $[m \times (r-l)]$ and $G_{k+1,k}$ $[m \times p]$ are transfer matrices of state, of control, and of perturbations, respectively, obtained by means of the following straightforward transformations:

$$\Phi_{k+1,k} = A^{-1}C;\; F_{k+1,k} = A^{-1}D;\; G_{k+1,k} = A^{-1}G. \tag{5}$$

It should be noted that the matrix A is the finite-difference equations of heat conduction is not singular ($\det(A) \neq 0$, since $[a_{ii}] \neq 0$, $i = 1, 2, \ldots, m$), and, consequently, it possesses an inverse matrix.

The recurrence algorithm for the discrete case of the Kalman filter is written as [6]

$$\hat{\mathbf{X}}_{k+1/k+1} = \hat{\mathbf{X}}_{k+1/k} + k_{k+1}\,[\mathbf{V}_{k+1} - H_{k+1}\hat{\mathbf{X}}_{k+1/k}], \tag{6}$$

where $\hat{X}_{k+1/k+1}$ $[m \times 1]$ is the estimates of the extended state vector obtained from the measurement vector $\hat{V}_{k+1}$ $[n \times 1]$ and from the prediction of the estimate of the extended state vector from preceding step to the one in question $\hat{X}_{k+1/k}$ $[m \times 1]$; H_{k+1} $[n \times m]$ is the measurement matrix which implements the link between the measurements vector and the extended state vector.

The prediction of the extended state vector $\hat{X}_{k+1/k}$ $\{\hat{T}_{k+1/k};\ \hat{\alpha}_{k+1/k}\}$ is found from the equations

$$\hat{\mathbf{T}}_{k+1/k} = \Phi^1_{k+1,k}\,\hat{\mathbf{T}}_{k/k} + F^1_{k+1,k}\,\mathbf{U}^1_k, \tag{7}$$

$$\hat{\alpha}_{k+1/k} = \hat{\mathbf{f}}\,(\tau_k), \tag{8}$$

while the weight matrix K_{k+1} $[m \times n]$ is found from the equation

$$K_{k+1} = P_{k+1/k}\,H^T_{k+1}\,[H_{k+1}P_{k+1/k}\,H^T_{k+1} + R_{k+1}]^{-1}. \tag{9}$$

In the above one has

$$P_{k+1/k} = \Phi_{k+1,k}\,P_{k/k}\,\Phi^T_{k+1,k} + G_{k+1/k}\,Q_k G^T_{k+1,k}; \tag{10}$$

$$P_{k/k} = P_{k/k-1} - K_k H_k P_{k/k-1}; \tag{11}$$

$\Phi_{k+1,k}$, $\Phi^1_{k+1,k}$, $F^1_{k+1,k}$, and $G_{k+1,k}$ are transfer matrices; $P_{k+1/k}$, $P_{k/k}$, Q_k, and R_{k+1} are the matrices of the prediction errors, estimation errors of the state vector, random perturbations at the system input, and measurement errors, respectively.

The transfer matrices $\Phi_{k+1,k}$ and $G_{k+1,k}$ can be found by using Eq. (1) and the matrices $\Phi^1_{k+1,k}$ and $F^1_{k+1,k}$ by using (3).

One of the most involved problems related to the use of the proposed filtering algorithm is the discrepancy between the estimated and the true values, which results in the lowering of the correcting effect of the subsequent measurements. Since the actual form of Eq. (2) when identifying the heat-transfer coefficients proves to be indeterminate, the determination of the predicted value of the vector $\boldsymbol{\alpha}$ and of the corresponding components of the matrix $\Phi_{k+1,k}$ is also impossible.

If α = const, then (2) can be replaced by the equation $\alpha = 0$; consequently, expression (8) assumes the form

$$\hat{\alpha}_{k+1/k} = \hat{\alpha}_{k/k}. \tag{12}$$

In the case $\boldsymbol{\alpha} = f(\tau)$ the following approach is proposed for obtaining an estimate prediction of the identification parameter $\boldsymbol{\alpha}$.

At the first time steps the predicted value of the state vector (within the estimated parameter $\boldsymbol{\alpha}$) is evaluated by using the formula (12), and then, through the two new estimated values of this parameter – $\hat{\alpha}_{k/k}$ and $\hat{\alpha}_{k+1/k+1}$ – a straight line is drawn, which represents the geometrical locus of the points of the predictable values of the vector for the next few steps or even until the procedure has ended. In the first alternative one selects again in a few (usually five to six) steps two consecutive values of the predicted parameters and another prediction straight line is drawn through them, etc.

One should select the value $\hat{\alpha}_{k/k}$ as the first prediction point; this corresponds to the lowest value of the difference $\|V_k - H_k\hat{X}_{k/k}\|$. Then the second point through which the prediction line will be drawn is the value $\hat{\alpha}_{k+1/k+1}$; Eq. (8) is now transformed into

$$\hat{\alpha}_{k+i+1/k+i} = M\tau_{k+i+1} + N,\ i = 1,\ 2, \ldots,\ s, \tag{13}$$

where s is the number of time steps on which a given straight line is operational; M and N are coefficient matrices dependent on $\hat{\alpha}_{k+1/k+1}$, $\hat{\alpha}_{k/k}$, and the time step. The finite-difference approximation of Eq. (13) is also taken into account when forming the transfer matrices.

A one-dimensional inverse problem of nonstationary heat conduction is considered as an example of identifying time-varying coefficients of heat transfer; its solution enables one to carry out a simultaneous reconstruction of the temperature field from limited data on the latter side by side with finding the relation $\alpha = f(\tau)$ (for known T_{av}).

When solving the above, the results of solving a direct heat-conduction problem were used as standard volume of the temperature for a flat wall made from a material with thermal characteristics dependent on time (λ = 50–0.03T (W/m·deg); $\alpha = 1.23 \cdot 10^{-5} - 1.05 \cdot 10^{-3}$ T (m²/sec)] and with the boundary conditions of the third kind. Two different laws for changing $\alpha = f(\tau)$ were considered; the results of the inverse problem were eventually compared with the latter:

$$\text{I.}\quad \alpha = \frac{50 - 0.03\, T_b}{L}\left[1 + 0.5\,\frac{(1.23 - 0.00105\, T_b)\cdot 10^{-5}\tau}{L^2} - \right. \tag{14}$$

$$\left. - 0.5 \exp\left[-(1.23 - 0{,}00105\, T_b)\cdot 10^{-5}\tau/L^2\right]\right.,$$

$$\text{II.}\quad \alpha = \frac{50 - 0.03\, T_b}{L}\exp\left[\frac{-(1.23 - 0.00105\, T_b)\cdot 10^{-5}\tau}{10L^2}\right]. \tag{15}$$

Since the heating of a wall from the initial state assumed to be $X_i(0) = 0°C$ was carried out symmetrically, the wall thickness equal to 0.08 m was broken in half, and the problem was now solved for a plate of thickness L = 0.04 m under the above-described boundary conditions of the third kind, $\alpha[T - T_{av}] = -\lambda dT/dn$ on one of the boundaries and on the vanishing boundary conditions of the second kind, and $\partial T/\partial n = 0$ on the other boundary. The temperature of the heating medium was adopted as 600°C. The thickness L was subdivided into five portions (the space step h = 0.01 m inside the plate, while h = 0.005 m at the boundaries). The time step was assumed to be $\Delta\tau$ = 30 min.

The temperature of the first point was adopted as the measured parameter, and the matrix H_{k+1} was obtained in agreement with the latter.

The initial estimate of the state vector was adopted arbitrarily $\hat{X}_{0/0}^T = [-50, -50, \ldots, -50, \hat{\alpha}_{0/0}]$, $\hat{\alpha}_{0/0}$ was adopted equal to 200 or to 500, and the covariance matrix of errors in the initial estimates was assumed to be equal to $P_{0/0} = c^2E$, where E is the identity matrix and c is a suitably high coefficient.

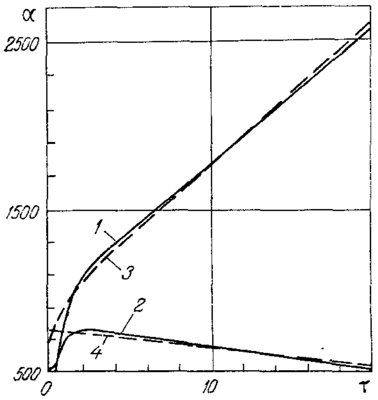

Fig. 1. The convergence of the estimates for identification parameter **α**: 1) $\alpha_{0/0}$ = 200 W/m²·deg; 2) $\alpha_{0/0}$ = 500 W/m²·deg; 3, 4) standard curves; τ) time, h.

The results of determining both relations $\alpha = f(\tau)$ are shown in Fig. 1 (curves 1 and 2) where for comparison the standard curves 3 and 4 are also shown; the latter correspond to the adopted relations $\alpha_I(\tau)$ and $\alpha_{II}(\tau)$ when solving the direct problem. The estimating curve 1 was obtained under the condition that the prediction straight line was revalued after every five steps starting from the fifth one; the curve 2 was obtained by again finding the prediction straight lines revalued after every five steps, but now, starting from the seventh step. As far as the reconstruction of the temperature field is concerned, the difference between the estimated temperature and the standard ones does not exceed 1-1.5% at the sixth-eighth step.

Thus, the obtained results enable one to conclude that the estimated values $\hat{\alpha}_{k/k}$ converge rapidly to the standard ones with a sufficiently free choice of the initial values $\hat{\alpha}_{0/0}$.

NOTATION

A, C, D, E, G, M, N	are the matrices;
A^T	is the transposed matrix;
A^{-1}	is the inverse matrix;
X	is the extended state vector;
T	is the temperature field vector;
α	is the vector of unknown parameter;
$\hat{\mathbf{X}}_{k+1/k+1}$, $\hat{T}_{k+1/k+1}$, and $\hat{\alpha}_{k+1/k+1}$	are the estimates of state vector;
$\hat{X}_{k+1/k}$, $\hat{T}_{k+1/k}$, $\hat{\alpha}_{k+1/k}$	are the predictions of state vectors;
$\Phi_{k+1,k}$, $\Phi^1_{k+1,k}$, $F^1_{k+1,k}$, and $G_{k+1,k}$	are the transition matrices;
$P_{k+1/k+1}$, $P_{k+1/k}$, Q_k, and R_{k+1}	are the covariance matrices;
K_{k+1}	are the weight matrices;
H_{k+1}	is the measurement matrix;
λ	is the heat-conduction coefficient;
τ	is the time;
h	is the grid step.

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PARAMETRIC METHOD OF SOLVING A NONLINEAR HEAT-CONDUCTION PROBLEM FOR A SEMIINFINITE BODY

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A new method is proposed for solving heat-conduction problems with nonlinear boundary conditions.

A very essential shortcoming of the well-known integral methods for solving nonlinear heat-conduction problems [1] is the a priori choice of the family of temperature profiles or of heat-flux density. The degree of approximation of the adopted distribution of the sought values to the true one and thus the error of the method depends one one's intuition; as a rule, they are only satisfactory in a finite range of the values of the parameter.

Up to the mid-1960's a similar situation could be observed as regards the related problem of evaluating the laminar boundary layer when the multiparameter method developed by Loitsyanskii [2] was published, showing the way for obtaining the families of the velocity profiles in the boundary-layer section in a rational manner. It was based on solving the boundary-layer differential equation in new dimensionless parameters (the similarity parameters), thus ensuring good accuracy of the obtained results when analyzing specific problems.

In this article an attempt is made to generalize the concepts of the Loitsyanskii method to the nonlinear problems of heat conduction.

We now consider a heat-conduction problem for a semiinfinite body with constant thermal characteristics, which can be formulated as follows:

$$\frac{\partial T(x, \tau)}{\partial \tau} = a \frac{\partial^2 T(x, \tau)}{\partial x^2}; \tag{1}$$

$$\begin{cases} -\lambda \dfrac{\partial T}{\partial x} = Q(T_p \;\; T_s, \tau) \text{ for } x = 0, \\ \dfrac{\partial T}{\partial x} = 0, \; T = T_\infty \text{ as } x \to \infty; \end{cases} \tag{2}$$

$$T = T_0(x) \text{ for } \tau = 0. \tag{3}$$

In this form, the problem (1)-(3) is referred to according to the classification of [1] as a nonlinear problem of the second kind, where the nonlinearity appears only in the boundary conditions (2).

Instead of the variable $T(x, \tau)$, the variable $q(x, \tau)$ is introduced by means of the relation

$$q(x, \tau) = -\lambda \frac{\partial T(x, \tau)}{\partial x}, \tag{4}$$

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